MATHEMATICS DISCUSSIONS:

Expectatio

Various strategies can help you build a classroom environment rich with mathematical discussion.

George J. Roy, Sarah B. Bush, Thomas E. Hodges, and Farshid Safi

Reflect on a time when your students were engaged in a mathematics classroom discussion. Ask yourself:

- Who was doing the majority of the talking?
- What was your role as the teacher during the dialogue?
- How did you facilitate student-to-student dialogue?
- How did you use this opportunity to formatively assess student understanding?

Strategic and carefully facilitated classroom discussions can foster a deep understanding of mathematics because—

> discourse in the mathematics classroom gives students opportunities to share ideas and clarify understandings, construct convincing arguments regarding why and how things work, develop a language for expressing mathematical ideas, and learn to see things from other perspectives. (NCTM 2014, p. 29)

Copyright © 2017 The National Council of Teachers of Mathematics, Inc. www.nctm.org.
All rights reserved. This material may not be copied or distributed electronically or in any other format without written permission from NCTM.

ns Matter

It is vital that students engage in such experiences in purposeful ways. However, in a more traditional setting, some students have experienced success in mathematics in other ways, such as by reviewing their homework, "checking" for right answers, and volunteering in class.

As we consider what it means for our students to "do mathematics" successfully, we must explore our role as teachers in a way that strategically supports students as powerful agents for their own learning. As we foster a classroom environment based on agreed-on expectations, we must help students learn how to engage in whole-class discussions that create opportunities for learning among all students (Stephan and Whitenack 2003).

Smith and Stein (2011) identified five instructional practices rooted in students' thinking that guide teachers in making important mathematical ideas public. In concert with engaging in the five practices, it is essential for a teacher to foster a classroom environment in which sense making remains central to daily learning experiences. One strategic way to build a classroom environment rich with mathematical discussions is to engage students in mathematical tasks that require them to build their understanding on the reasoning of others (Knudsen et al. 2014).

As students discuss a task, the teacher's role is to continually work with them to negotiate what high-quality discourse sounds like. For example, it is important to establish student expectations, such as that all students will—

1. participate, even those who do not raise their hands;
2. explain and justify their thinking;
3. restate a classmate's reasoning;
4. make sense of another classmate's reasoning; and
5. ask a question if they are not sure that they understand.

These expectations encourage active listening among students while maximizing the opportunity for all students to participate in the discussion (Brooks and Dixon 2013; Chapin, O'Connor, and Anderson 2009; Stephan and Whitenack 2003). Although the negotiation of classroom expectations develops over time before being sustained by students (Dixon, Andreasen, and Stephan 2009), these expectations provide the requisite foundation for students' interactions during a meaningful whole-class discussion.

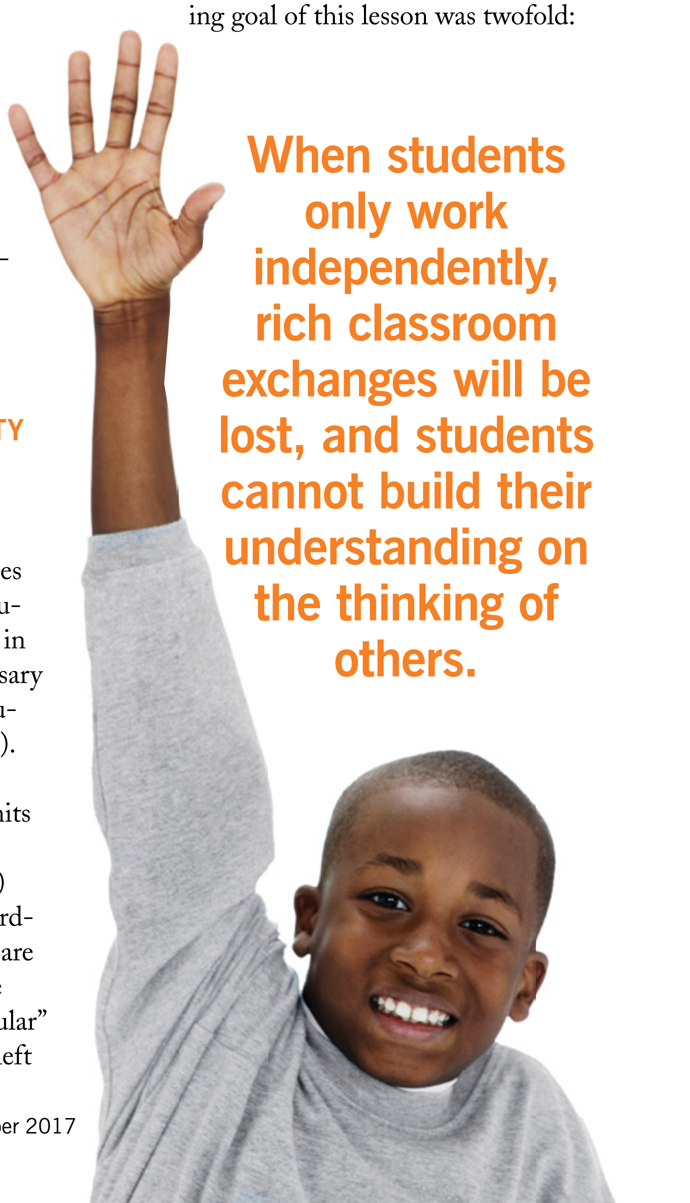
In our experience, getting started can often be the most challenging part of facilitating productive classroom dialogue. Consequently, in this article, we share an initial classroom episode in which classroom expectations were introduced to create an environment for all students to engage in meaningful mathematics dialogue.

A CLASSROOM CONVERSATION: THE DISTRIBUTIVE PROPERTY

Arithmetic to Algebra: Leveraging Prior Knowledge

Our classroom episode focuses on the concept of the distributive property. Some students in the middle grades lack necessary understanding of the distributive property (Edwards 2000). Often, the way the distributive property is presented limits students' understanding. For example, Ding and Li (2010) point out that strategies regarding the distributive property are mainly presented with whole numbers and follow the "regular" direction of expansion from left to right. Another root of this problem can be traced back to a general lack of understanding of properties of arithmetic. Students' understandings of the properties of arithmetic are greatly enhanced in the middle grades if algebraic connections are explicitly connected to arithmetic properties previously explored in elementary school. As Carpenter, Franke, and Levi (2003, p. 2) surmise, "The fundamental properties that children use in carrying out arithmetic calculations provide the basis for most of the symbolic manipulation in algebra."

In the following classroom episode, the first author explored equivalent expressions using the distributive property with a class of 24 sixth-grade students. The mathematical learning goal of this lesson was twofold:



When students only work independently, rich classroom exchanges will be lost, and students cannot build their understanding on the thinking of others.

PREVIOUS PAGE: GRADYREESE/ISTOCK; THIS PAGE: BRAND X PICTURES/THINKSTOCK

First, by the end of the lesson the students would be able to leverage their procedural knowledge of the distributive property to recognize and articulate the understanding that algebraic expressions in different forms can be equivalent (NCTM 2006). Second, students would explore equivalent forms of an algebraic expression and discover a generalization of the distributive property (Carpenter, Franke, and Levi 2003). The National Research Council (2001) has described students' proficiency with mathematical ideas as a combination of five interwoven strands, including both procedural fluency and conceptual understanding. We sought to leverage the procedural knowledge of the students to construct more conceptual understandings, which would eventually lead to greater proficiency with the distributive property through classroom-level discourse. We hope this episode will serve as a model for other middle school mathematics teachers as they embark on their own journey toward supporting their students as they move toward purposefully engaging in whole-class discussions.

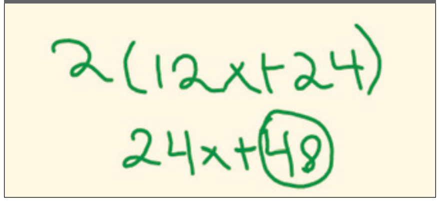
Beginning the Conversation

The teacher began by introducing a clear set of expectations that the class would follow when investigating the task. The following dialogue represents a portion of that conversation.

Teacher: One thing that we may do differently is that I will call on you even if your hand is not raised. [Students laugh.]
Aubrey: Crazy!
Teacher: . . . Why do I do that?
Sydney: That way we can learn from our mistakes.
Teacher: Why else?
Aubrey: Because if a person is not paying attention, you know they are not paying attention.
Cameron: Because someone else may not understand and have a question.
Teacher: Jordan. . . . What did your classmates say?
Jordan: Some of us may have questions.
Teacher: Good; someone may have a question. What else did they say?
Izzy: Kind of like a warning that you [teacher] are like watching them.
Shawn: To make sure that we are paying attention.

Although some students thought that a major reason the teacher would call on them was to maintain classroom management, this was not the case. The teacher intentionally introduced the expectation that *all* students would participate in the class discussion—not just the students who raised their hands—so that each student's mathematical reasoning could be included. Furthermore, when the teacher called on Jordan to rephrase her peers' thinking, the teacher initiated the expectation that each student would listen actively to what his or her peers were explaining. Participation by *all* students and active listening were two of the classroom expectations established, and their significance cannot be overstated. Establishing these key expectations creates an environment in which students can support one another's thinking. This environment supports a reflexivity in which individuals contribute to the mathematical understandings of the class while, at the same time, the whole class impacts the ways an individual thinks about the mathematics (Cobb and Yackel 1996). Furthermore, it allows the teacher to call on students who may have a mathematical contribution but feel that their status is not high enough to contribute during whole-class discussion (Civil and Planas 2004). The emphasis on these two expectations resulted in all students in the class contributing to the discussion during the lesson. After the teacher had introduced and discussed the expectations, students explored the following task:

Fig. 1 Lynn proposed this expansion of $2(12x + 24)$.



> Sofia thinks $2(12x + 24)$ can be rewritten as $6(4x + 8)$, but Andre thinks it cannot.
>
> - Who is right? How do you know?
> - What are other equivalent ways of writing $2(12x + 24)$?

The initial question in the task was meant to leverage students' procedural knowledge of the distributive property and serve as a foundation from which the students would eventually generalize the property. To accomplish this goal, we gave students individual think time and then prompted them to share their thinking with a shoulder partner. This pair dialogue was an essential step in creating an environment in which students felt comfortable sharing their thinking. It simultaneously reinforced the expectation that students ought to be prepared to explain and justify their reasoning as well as the expectation that they make sense of another's reasoning. Student-to-student dialogue is fundamental in a whole-class discussion. After the students quickly established that Sofia was correct, the teacher sparked the whole-class discussion by inviting individuals to share their initial thoughts regarding other equivalent ways the expression could be written (see **fig. 1**).

This allowed the teacher to monitor and sequence (Smith and Stein 2011) reasoning by allowing students to publicize their initial conceptions

Fig. 2 Ali was the first to propose an equivalent expression with a negative coefficient.

$$-2(-12x + -24)$$

Fig. 3 A student suggested a decimal coefficient after discovering that equivalent expressions involving negative coefficients were possible.

$$32(0.75x + 1.5)$$

of equivalence by connecting their thinking with the reasoning of others. This is shown in the following portion of the whole-class discussion. Note the manner in which the teacher facilitated the discussion by framing the importance of sense making through student reasoning.

Teacher: Let's talk about it . . . I am going to have you make sense of what Lynn is doing. [See **fig. 1**.] . . . Nico, tell us what you think she did.
Nico: I think she multiplied 2 by 12 and 2 by 24.
Teacher: Did she just multiply 2 by 12?
Student: She distributed the 2 to the 12 and 24.
Pat: You do 2 times 12*x*.
Shawn: I have a question. . . . Don't we need to do what's in the parentheses first before multiplying by 2?
Teacher: Let's talk about that. Don't you need to do what's in the parentheses first? You know the rule that parentheses come first.
Student: I don't think. You can't do what is in parentheses first because you can't add 12*x* plus 24.
Jamie: They are not like terms.

This exchange created the opportunity to ground the whole-class conversations regarding equivalent expressions to the students' procedural understanding of the distributive property, as recommended by Lloyd, Herbel-Eisenmann, and Star (2011). Moreover, it supported the view that mathematics learning should focus on developing and connecting concepts and procedures by having students reason and engage in discourse (NCTM 2014), while also providing the members of the class with an opportunity to question a rigid interpretation of the order of operations (Karp, Bush, and Dougherty 2015). By engaging in classroom dialogue facilitated by the teacher, the *students* made sense of like terms and the order of operations as they explored the task focused on the distributive property.

As the whole-class dialogue continued, we anticipated that students would correctly distribute whole number quantities to each term in the parentheses; however, we wanted students to move beyond this procedural understanding to reason more flexibly and consider other equivalent ways that the algebraic expression could be written. Consequently, in the next part of the task, students were challenged to generate expressions equivalent to $2(12x + 24)$. The following portion of whole-class dialogue consistent with the goals and expectations of the teacher showcases a student's response to the task and how other students in the class made sense of her reasoning.

Ali: Let's see, $-2(-12x + -24)$. [See **fig. 2**.]
Teacher: We are going to have to stop for a second, because Cam got really excited.
Cam: I did.
Teacher: Why did you get excited?
Cam: 'Cause I didn't think of a negative number.
Teacher: You didn't think of a negative number. Do you think it works?
Students: Yes.
Cam: I do! [Students laugh.]

Many of the students expressed the same excitement when negative coefficients were suggested because the idea opened the door to an entirely new way of thinking about this task. At this point, if the expectation to make one's thinking public had not been introduced at the beginning, Cam might have continued to hold on to a myopic understanding of the property. Moreover, when students publicize their thinking in conversations such as this one, the teacher and students are able to assess student thinking in a nonthreatening and inviting way that results in expanded thinking that is a valuable source of sense making. The brief dialogue also demonstrates how students must be active listeners to what their classmates are saying and be ready to react and contribute to each other's learning during a classroom discussion. Although students often want to default to their own thought process, it is vital that we as teachers make this expectation explicit—active listening is at the core of students' ability to refine their understanding of mathematics through examining the thinking of others.

As the conversation continued, another student then suggested the decimal expansion shown in **figure 3**, which became the impetus for students to explore equivalent expressions in a more generalized way. This new situation led to the following dialogue, which highlights students' investigation of equivalence.

Teacher: Corey, tell us about this "crazy" one.
Corey: It is not really one that people normally think of.
Teacher: Why is that?
Corey: Because when most people when they see decimal points, they think it is a lot harder than just whole numbers.
. . .

PHOTODISC/THINKSTOCK

Kerry: Once I saw a decimal number, I figured, yes, that could be right because 32 is bigger than 12 but if you multiply it by a decimal, it is going to get smaller; that would allow it to become 24.

Fran: Thirty-two is bigger than 12 just like Kerry said we need to multiply it by a number less than 1 to make it smaller . . . three-fourths of 32 is 24. . . .

Teacher: Justice, help him out. You said you do not need a calculator.

Justice: You really don't 'cause kind of what Fran said, 3 times 32 is 96 and then divide it by 4, which equals 24 plus one and one-half make that improper and then you multiply. . . .

Pat: I think that you could do the same thing that you did with .75 . . . you could do it an easier way. You know that 1 times 32 so split 32 in half and add half of 32 and 32 together. . . .

Kerry: I have another one; 64 times $(.375x + .75)$. [See **fig. 4**.]

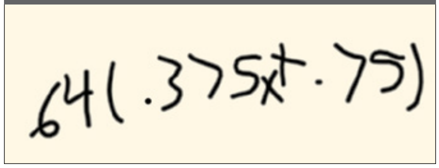
Teacher: Which one of the other equivalent expressions is this like? Bailey?

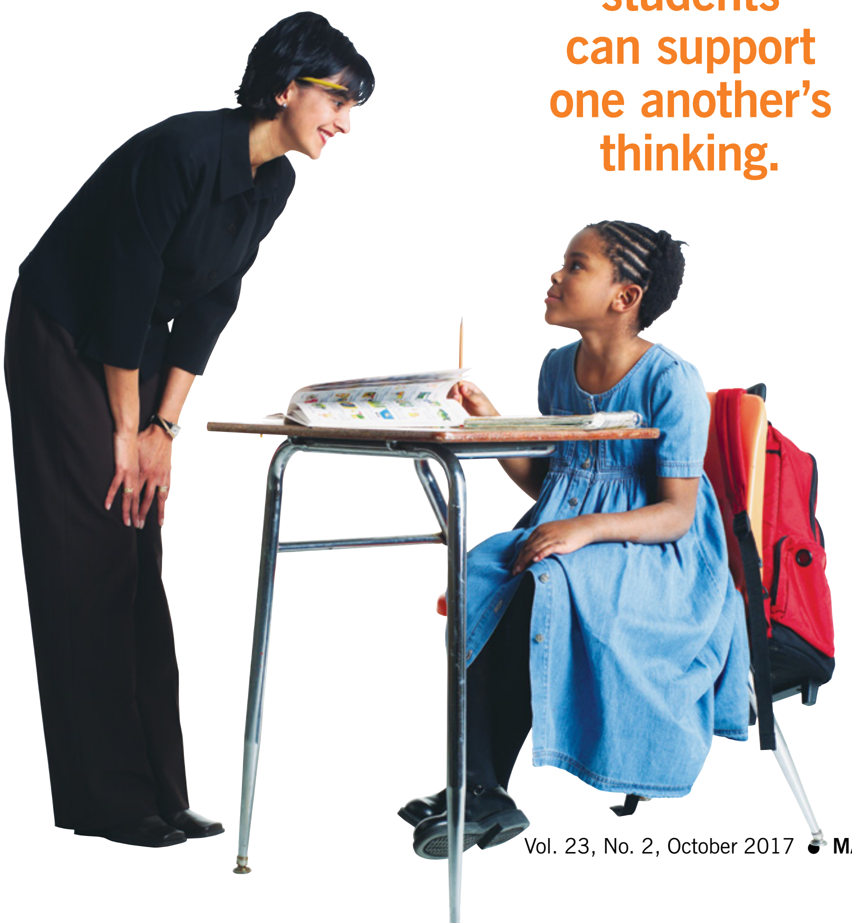
Bailey: $[32(0.75x + 1.5)]$ They multiplied the 32 by 2 to get 64 and then divided the numbers inside the parentheses by 2.

When students only work independently, classroom exchanges such as these will be lost, and students cannot build their understandings on the thinking of their classmates. For example, it is important that Fran made sense and rephrased using a decimal coefficient more precisely than Kerry's assertion. If this had not happened, some students might have thought that multiplying any decimal including those greater than 1 would yield a smaller product. This is why we argue that the best sense making takes place during a whole-class discussion. In this case, the whole-class dialogue regarding the idea of multiplying the factor outside the parentheses while dividing the terms inside the parentheses was vital to support students' conceptual understanding of the mathematics underlying algebraic generalization (Carpenter, Franke, and Levi 2003). During the lesson, it became clear that discussing the task in this way (rather than students only working independently) allowed the students to support one another and elevated their thinking to a whole new level.

Establishing key expectations creates an environment in which students can support one another's thinking.

Fig. 4 Kerry suggested a decimal equivalent expression.





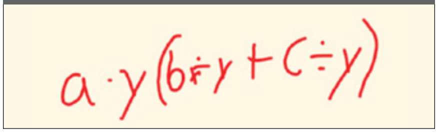
THE BIG MOMENT: ARRIVING AT A GENERALIZATION

At the same time, we were interested in leveraging students' ability to generate equivalent expressions and pushing them toward articulating a generalization of the distributive property. To address this goal, the teacher challenged the students to make connections among equivalent expressions. The connections were then discussed to determine how the equivalent expressions were related.

Teacher: [To the class] I am very impressed with your thinking. Carolyn, start us out with your group's thinking.

Carolyn: I said they are bigger and smaller versions of each other, and Emerson told me that all of the numbers in the expressions are factors of 24 and 48.

Fig. 5 Hope proposed this algebraic generalization for equivalent expressions.



Mackenzie: There are infinite number of ways [to write equivalent expressions].

Teacher: Hope? You have a thought?

Hope: Yes, I kind of have a formula. [See **fig. 5**.] Every time you multiply the outside number, you divide the inside numbers because if you multiply both of them, the number will go over what you need, and you need to keep it. . . .

Teacher: So you [students] agree with her then? Cameron?

Cameron: If you divide or multiply the number outside the parentheses, you use the inverse operation for the numbers inside the parentheses.

Teacher: What did Cameron just say?

Jamie: I honestly don't know.

Teacher: Then ask him a question.

Jamie: Can you repeat that please?

Cameron: Whatever you do to the number outside the parentheses, either multiply or divide, you use the inverse operation for whatever is inside the parentheses.

Jamie: OK.

Teacher: What are you okaying?

Jamie: I understand now.

Teacher: What do you understand?

Jamie: OK, if you do multiplication on the outside, you'll have to do division on the inside, but if you divide on the outside, you'll have to multiply on the inside.

Getting students to discuss this abstract understanding of the property, where students formulate a rather complex algebraic generalization, was one of the learning goals of our task. Initially, the class discussion began by concentrating on the students' procedural understanding of the distributive property. It evolved into a more sophisticated conversation that led to the generalization of the property. Furthermore, arriving at a generalization showcases how building student understanding of the distributive property through a meaningful progression can help students truly understand a mathematical concept—developing conceptual understanding and procedural fluency simultaneously. In our case, our progression began with a review of the distributive property arithmetically, followed by a large portion of the task dedicated to the discovery that there are an infinite number of equivalent expressions, and finally culminating in abstracting an algebraic generalization (Driscoll 1999).

After reflecting on the classroom implementation of this task, we determined that next time we will incorporate additional discussion questions throughout the duration of the classroom episode that will provide an additional formative lens into students' conceptual understanding. (See the **sidebar** on page 105 for some examples.)

THE SHIFT TO A CLASSROOM COMMUNITY

The classroom episode we shared showcases a needed shift from the teacher being the sole disseminator of mathematical knowledge to the entire classroom community taking responsibility for mathematical conversations. With this important shift, students realize that they are expected to be responsible for their own learning and to aid in the learning of their classmates. This new scenario also enables and encourages students to succeed in doing so under the guidance of their teacher.

We hope that our episode has provided a practical example of how to structure such an environment as well as strong evidence for the high level of understanding—in our case, creating an algebraic generalization—that students can reach.

BIBLIOGRPAHY

Brooks, Lisa A., and Juli K. Dixon. 2013. "Changing the Rules to Increase Discourse." *Teaching Children Mathematics* 20 (2): 84–89.

Carpenter, Thomas P., Megan Loef Franke, and Linda Levi. 2003. *Thinking Mathematically: Integrating Arithmetic and Algebra in Middle School*. Portsmouth, NH: Heinemann.

Chapin, Suzanne H., Catherine O'Connor, and Nancy Canavan Anderson. 2009. *Classroom Discussions: Using Math Talk to Help Students Learn*. 2nd ed. Sausalito, CA: Math Solutions.

Civil, Marta, and Núria Planas. 2004. "Participation in the Mathematics Classroom: Does Every Student Have a Voice?" *For the Learning of Mathematics* 24 (1): 7–12.

Cobb, Paul, and Erna Yackel. 1996. "Constructivist, Emergent, and Sociocultural Perspectives in the Context of Developmental Research." *Educational Psychologist* 31 (3–4): 175–90.

Ding, Meixia, and Xiaobao Li. 2010. "A Comparative Analysis of the Distributive Property in U.S. and Chinese Elementary Mathematics Textbooks." *Cognition and Instruction* 28 (2): 146–80.

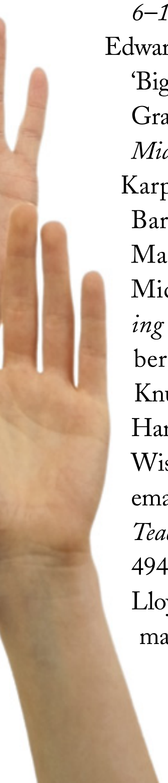
Students realize that they are expected to be responsible for their own learning.

PHOTKA/THINKSTOCK

Ask More Questions

Think about asking these questions as a further check on students' conceptual understanding:

1. When simplifying an expression involving the distributive property, could we just follow the order of operations—first simplifying what is in the parentheses before multiplying by the number outside the parentheses?
2. Why is it that when you have an expression such as $3(4x \bullet 5)$ you can simplify to $3(20x)$, then to $60x$, but you could not do that with our given problem?
3. What is a real-life scenario in which you would use the distributive property to solve a problem?
4. How could you pictorially represent the distributive property?



Dixon, Juli K., Janet B. Andreasen, and Michelle Stephan. 2009. "Establishing Social and Sociomathematical Norms in an Undergraduate Mathematics Content Course for Prospective Teachers: The Role of the Instructor." In *AMTE Monograph VI: Scholarly Practices and Inquiry into the Mathematics Preparation of Teachers*, pp. 43–66). San Diego, CA: Association of Mathematics Teacher Educators.

Driscoll, Mark. 1999. *Fostering Algebraic Thinking: A Guide for Teachers Grades 6–10.* Portsmouth, NH: Heinemann.

Edwards, Thomas G. 2000. "Some 'Big Ideas' of Algebra in the Middle Grades." *Mathematics Teaching in the Middle School* 6 (September): 26–31.

Karp, Karen S., Sarah B. Bush, and Barbara J. Dougherty. 2015. "12 Math Rules That Expire in the Middle Grades." *Mathematics Teaching in the Middle School* 21 (November): 208–15.

Knudsen, Jennifer, Teresa Lara-Meloy, Harriette Stallworth Stevens, and Daisy Wise Rutstein. 2014. "Advice for Mathematical Argumentation." *Mathematics Teaching in the Middle School* 19 (8): 494–500.

Lloyd, Gwendolyn, Beth Herbel-Eisenmann, and Jon Star. 2011. *Developing Essential Understanding of Expressions, Equations, and Functions for Teaching Mathematics in Grades 6–8.* Reston, VA: National Council of Teachers of Mathematics.

National Council of Teachers of Mathematics (NCTM). 2006. *Curriculum Focal Points for Prekindergarten through Grade 8 Mathematics: A Quest for Coherence.* Reston, VA: NCTM.

———. 2014. *Principles to Actions: Ensuring Mathematical Success for All.* Reston, VA: NCTM.

National Research Council (NRC). 2001. *Adding It Up: Helping Children Learn Mathematics*, edited by Jeremy Kilpatrick, Jane Swafford, and Bradford Findell. Washington, DC: National Academies Press.

Smith, Margaret Schwan, and Mary Kay Stein. 2011. *Five Practices for Orchestrating Productive Mathematics Discussions.* Reston, VA: National Council of Teachers of Mathematics.

Stephan, Michelle, and Joy Whitenack. 2003. "Establishing Classroom Social and Sociomathematical Norms for Problem Solving." In *Teaching Mathematics through Problem Solving: Prekindergarten–Grade 6*, edited by Frank K. Lester Jr., pp. 149–62. Reston, VA: National Council of Teachers of Mathematics.

ACKNOWLEDGMENT

The authors wish to thank Amy Berube, Crossroads Middle School, Columbia, South Carolina, as well as Lindsay Flanagan and Chelsey Whitman, who are University of South Carolina middle-school-level interns, for their classroom support.

George J. Roy, roygj@mailbox.sc.edu, is an associate professor at the University of South Carolina, having taught middle school math for eight years in Orange County Florida Public Schools. During his public school tenure, he achieved a NBPTS certification in Early Adolescence Mathematics. His current research efforts include examining uses of dynamic technology in middle school math classrooms. **Sarah B. Bush**, sarah.bush@ucf.edu, is an associate professor of K–12 STEM education at the University of Central Florida's School of Teaching, Learning, and Leadership in Orlando. She is a former middle-grades mathematics teacher who is interested in integrated, relevant, and engaging mathematics tasks. **Thomas E. Hodges**, hodgeste@sc.edu, is an associate professor and associate dean of academic affairs at the University of South Carolina. He enjoys helping students see and act on the world as mathematicians through his work with preservice and in-service teachers. **Farshid Safi**, farshid.safi@ucf.edu, is an assistant professor of mathematics education at the University of Central Florida. He focuses on developing teachers' conceptual understanding of K–12 mathematics, as well as connecting essential topics in professional development through the use of multiple representations and technology.